INTRODUCTION TO ALGEBRAIC NUMBER THEORY 2018 – MIDTERM EXAM

- *Questions are worth* 112 *points. You will be graded out of* 100 *(i.e. a score greater than* 100 *is treated as* 100*). In particular you can skip questions worth a total of* 12 *points.*
- *You are allowed one handwritten A4 sheet (both sides) in the hall.*

Problem 1 (4 points each)**.** Answer in True/False. If the statement is False, give a counter example (or a proof/explanation). If it is True, no proof is necessary.

(1) In any ring R the ideal (0) is a prime ideal.

(2) The prime ideals of the ring $\mathbb{Z}[i]$ are of the form (f) with $f \in \mathbb{Z}[i]$ where either $f = 0$ or f is an irreducible.
(Recall that an element $f \neq 0$ of a ring R is said to be an irreducible *if any decomposition $f = gh$ with $g, h \in R$ implies that one of g, h is a unit.)*

(3) Let K denote the field $\mathbb{Q}(X,Y)$ or the field $\mathbb{F}_p(X,Y)$. Let L be the finite extension $K(X^{\frac{1}{p}}, Y^{\frac{1}{p}})$. Then there is an $\alpha \in L$, such that $L = K(\alpha)$.

(4) An Artinian ring has finitely many prime ideals.
(Recall that a Noetherian ring is *said to be* Artinian *if all prime ideals are maximal).*

(5) Let R be a Dedekind domain, and I be a non-zero ideal of R. Then the fractional ideal J such that $I \cdot J = R$ is unique.

(6) Let $R \subset R'$ be a ring extension. Then the set $\{r \in R' \mid r \text{ integral over } R\}$ is a subring of R'.

(7) Let S be a subset of a ring R which is multiplicatively closed. Then the mapping $I \mapsto S^{-1}I$ from $\{\text{ideals } I \text{ of } R \text{ such that } I \cap S \text{ is empty}\}$ to $\{\text{ideals of } S^{-1}R\}$ is bijective.

(8) Let $\mathcal{O}_K$ denote the ring of integers of a number field K. Then it is possible that $\mathcal{O}_K \cap \mathbb{Q}$ is a strictly larger set than $\mathbb{Z}$.

Problem 2 (14 points)**.** Prove Nakayama's Lemma. That is, show the following: Let R be a local ring and the unique maximal ideal of R be denoted as $\mathfrak{m}$. Let M be a finitely generated R-module and $N \subset M$ a submodule such that $M = \mathfrak{m}M + N$. Then $N = M$.

Problem 3 (14 points)**.** Let L/K be an extension of number fields and let $\mathcal{O}_K$ respectively $\mathcal{O}_L$ denote ring of integers in K respectively L. For $x \in \mathcal{O}_L$ show that $N_{L/K}(x)$ is a unit in $\mathcal{O}_K$ if and only if x is a unit in $\mathcal{O}_L$.

Problem 4 (14 points)**.** Let $K = \mathbb{Q}(\sqrt{d})$ where d is not-necessarily square free. Compute the ring of integers $\mathcal{O}_K$ of K.

Problem 5 (14 points)**.** Let $\mathcal{O}_K$ denote the ring of integers in a number field K and $p \in \mathbb{Z}$ be a prime number. Let $p \cdot \mathcal{O}_K = \mathfrak{P}_1^{e_1} \cdots \mathfrak{P}_r^{e_r}$ where $\mathfrak{P}_i \subset \mathcal{O}_K$ denotes prime ideal such that $e_i \in \mathbb{Z}$ and $e_i > 0$. Show that $\{\mathfrak{P}_1, ..., \mathfrak{P}_r\}$ is precisely the set of prime ideals $\mathfrak{Q}$ such that $\mathfrak{Q} \cap \mathbb{Z} = (p)$.

Problem 6 (12 points each)**.** Let L be a number field and $\mathcal{O}_L$ be its ring of integers.

(1) Let $x \in \mathcal{O}_L$ be non-zero. Show that $\frac{N_{L/\mathbb{Q}}(x)}{x} \in \mathcal{O}_L$.

(2) Let L be a number field and $a \in \mathbb{Z}$. Let X be the set of ideals $\{(x) \subset \mathcal{O}_L \mid N_{L/\mathbb{Q}}(x) = a\}$. Show that $|X| \leq |\mathcal{O}_L/(a)|$.